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ON THE STABILITY OF LINEAR SYSTEMS WITH RANDOM PARAMETERS

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We propose an approximate method for the investigation of the stability of systems of linear equations with stationary random coefficients, based on the use of the method of perturbations. The problem is reduced to the investigation of the stability of a system of finite-difference equations whose coefficients are determined by the spectral densities of the random parameters. Stability conditions for systems of linear equations with random coefficients have been considered by many authors [1 - 5]. For systems whose coefficients are Gaussian white noises, exact stability criteria have been obtained [1]. Approximate conditions based on the use of asymptotic methods have been found in a number of papers [3, 4] principally for second-order systems with small stationary perturbations of the parameters. The application of these same methods to higher-order systems leads to complicated calculations.

1. We consider the nth order equation

$$\mathbf{y}^{\cdot} = [C + \mu G(t) - \mu^2 B_1]\,\mathbf{y} \tag{1.1}$$

Here C is a real $n \times n$ matrix with eigenvalues $\lambda_s = ik_s$ $(s = 1,\ldots, 2r)$, $\operatorname{Re}\lambda_s < 0$ $(s = 2r + 1,\ldots, n)$; we assume that all the k_s are distinct (and, obviously, pairwise opposite). The elements of the matrix $G(t)$ are centered stationary random processes, B_1 is a real matrix, μ is a small parameter.

If $\mathbf{h}_m$ $(m = 1,\ldots, n)$ are the linearly independent eigenvectors of matrix C, then the transformation $\mathbf{x} = H^{-1}\mathbf{y}$, where $H = \|\mathbf{h}_1,\ldots, \mathbf{h}_n\|$, reduces system (1.1) to the form

$$\mathbf{x}^{\cdot} = [A + \mu P(t) - \mu^2 B]\,\mathbf{x} \tag{1.2}$$

where $A = \|k_1 i,\ldots, k_{2r} i, \lambda_{2r+1},\ldots, \lambda_n\|$ is a diagonal matrix and $P = H^{-1}GH$ and $B = H^{-1}B_1H$ are complex matrices. Using the ideas of the method of perturbations, we seek the solution of Eq. (1.2) in the series form

$$\mathbf{x} = \mathbf{x}_0 + \mu\mathbf{x}_1 + \mu^2\mathbf{x}_2 + \ldots \tag{1.3}$$

under the initial conditions $\mathbf{x}_0(0) = \mathbf{x}(0) = \mathbf{c}$, $\mathbf{x}_j(0) = 0$ $(j > 0)$, by assuming that for each realization of $P(t)$ this series converges on some interval $[0, T]$, where $T \sim \mu^{-1}$. Substituting (1.3) into (1.2) and separating out the terms with different powers of μ, we obtain a system of recurrence equations

$$\mathbf{x}_0^{\cdot} = A\mathbf{x}_0, \qquad \mathbf{x}_1^{\cdot} = A\mathbf{x}_1 + P(t)\,\mathbf{x}_0, \qquad \mathbf{x}_2^{\cdot} = A\mathbf{x}_2 + P(t)\,\mathbf{x}_1 - B\mathbf{x}_0 \tag{1.4}$$

solving which we find

$$\mathbf{x}_0 = e^{At}\mathbf{c}, \qquad \mathbf{x}_1 = \int_0^t e^{A(t-\tau)} P(\tau)\, e^{A\tau}\mathbf{c}\, d\tau$$

$$\mathbf{x}_2 = \int_0^t e^{A(t-\tau)}\left[P(\tau)\int_0^\tau e^{A(\tau-\tau_1)} P(\tau_1)\, e^{A\tau_1}\mathbf{c}\, d\tau_1 - Be^{A\tau}\mathbf{c}\right] d\tau \tag{1.5}$$

We shall judge the stability of system (1.2) from the variation of the mean value of the square of the norm of $\mathbf{x}(t)$, $\langle \mathbf{x}(t)\,\bar{\mathbf{x}}(t)\rangle$, where the bar on top denotes the conjugate quantity. Since

$$\langle \mathbf{x}(t)\,\bar{\mathbf{x}}(t)\rangle = \sum_{s=1}^{n} \langle x_s(t)\,\bar{x}_s(t)\rangle \tag{1.6}$$

the problem is reduced to computing the mean-square values of $|x_s(t)|$. From (1.3) we find

$$\langle |x_s|^2\rangle = \langle x_{0s}\bar{x}_{0s}\rangle + \mu\left(\langle x_{1s}\bar{x}_{0s}\rangle + \langle x_{0s}\bar{x}_{1s}\rangle\right) + \mu^2\left(\langle x_{1s}\bar{x}_{1s}\rangle + \langle x_{2s}\bar{x}_{0s}\rangle + \langle x_{0s}\bar{x}_{2s}\rangle\right) + \ldots \tag{1.7}$$

Let the components c_s of the initial conditions vector $\mathbf{c}$ be random variables satisfying the conditions

$$\langle c_s \bar{c}_m\rangle = \langle |c_s|^2\rangle\,\delta_{sm}, \qquad \langle p_{lk}(t)\,\bar{c}_s\rangle \equiv 0 \qquad (l, k, s, m = 1, \ldots, n) \tag{1.8}$$

where δ_{sm} is the Kronecker symbol, $p_{lk}(t)$ are the elements of matrix $P(t)$. In what follows, when computing $\langle \mathbf{x}(t)\,\bar{\mathbf{x}}(t)\rangle$ we shall carry out the averaging not only over the realizations of $P(t)$ but also over the set of initial conditions. If here the quantity $\langle \mathbf{x}(t)\,\bar{\mathbf{x}}(t)\rangle$ remains bounded as $t \to \infty$, whatever be the set c_s satisfying conditions (1.8), we take system (1.2) as being stable. By computing the terms occurring in (1.7) and taking (1.8) into account, we obtain

$$\langle x_{0s}(T)\,\bar{x}_{0s}(T)\rangle = \langle e^{\lambda_s T} c_s e^{\bar{\lambda}_s T}\bar{c}_s\rangle = \langle |c_s|^2\rangle\, e^{2\operatorname{Re}\lambda_s T} = e^{2\operatorname{Re}\lambda_s T}\langle x_s(0)\,\bar{x}_s(0)\rangle \tag{1.9}$$

$$\langle x_{0s}(T)\,\bar{x}_{1s}(T)\rangle = \int_0^T e^{\lambda_s T} e^{\bar{\lambda}_s(T-\tau)} \sum_{m=1}^{n} e^{\bar{\lambda}_s\tau}\langle \bar{p}_{sm}(\tau)\, c_s\bar{c}_m\rangle\, d\tau = 0 \tag{1.10}$$

$$\langle x_{1s}(T)\,\bar{x}_{1s}(T)\rangle = e^{2\operatorname{Re}\lambda_s T}\sum_{m=1}^{n}\langle |c_m|^2\rangle \int_0^T\int_0^T e^{(\lambda_m-\lambda_s)\tau} e^{(\bar{\lambda}_m-\bar{\lambda}_s)\tau_1} K_{sm}(\tau-\tau_1)\, d\tau_1\, d\tau =$$

$$= e^{2\operatorname{Re}\lambda_s T}\sum_{m=1}^{n}\langle x_m(0)\,\bar{x}_m(0)\rangle \frac{1}{2\pi}\int_0^\infty S_{sm}(\omega)\,|e^{(\lambda_m-\lambda_s+i\omega)T} - 1|\,|\lambda_m - \lambda_s + i\omega|^{-2}\, d\omega \tag{1.11}$$

$$\langle x_{2s}(T)\,\bar{x}_{0s}(T)\rangle = e^{2\operatorname{Re}\lambda_s T}\langle |c_s|^2\rangle\left\{\sum_{m=1}^{n}\int_0^T\!\!\int_0^{\tau} e^{(\lambda_m-\lambda_s)(\tau-\tau_1)}K'_{sm}(\tau-\tau_1)\,d\tau_1\,d\tau - b_{ss}T\right\} =$$

$$= e^{2\operatorname{Re}\lambda_s T}\langle x_s(0)\,\bar{x}_s(0)\rangle\left\{\sum_{m=1}^{n}\frac{1}{2\pi}\int_{-\infty}^{\infty} S'_{sm}(\omega)\left[\frac{e^{(\lambda_m-\lambda_s+i\omega)T}-1}{(\lambda_m-\lambda_s+i\omega)^2} - \frac{T}{\lambda_m-\lambda_s+i\omega}\right]d\omega - b_{ss}T\right\} \qquad (1.12)$$

Here

$$K_{sm}(\tau) = \langle p_{sm}(t)\,\bar{p}_{sm}(t-\tau)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{sm}(\omega)\,e^{i\omega\tau}\,d\omega$$

$$K'_{sm}(\tau-\tau_1) = \langle p_{sm}(\tau)\,p_{ms}(\tau_1)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} S'_{sm}(\omega)\,e^{i\omega(\tau-\tau_1)}\,d\omega$$

Suppose that the relation $\lambda_s - \lambda_m = ik_{sm}$, where k_{sm} is a real number, is fulfilled for some s and m Then singularities exist in the integrands in (1.11) and (1.12), leading when integrating to the appearance of terms proportional to T. To separate these terms we take it that $T \gg T_1$ (T_1 is the correlation time of the processes $p_{sm}(t)$); then the relations $S_{sm}(\omega) \approx S_{sm}(k_{sm}) = \text{const}$, $S'_{sm}(\omega) \approx S'_{sm}(k_{sm}) = \text{const}$ are valid for $S_{sm}(\omega)$ and $S'_{sm}(\omega)$ in the ranges $[k_{sm} - 1/T_1,\ k_{sm} + 1/T_1]$. Taking this into account, we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{sm}(\omega)\,|e^{(\lambda_m-\lambda_s+i\omega)T} - 1|^2\,|k_{sm}-\omega|^2\,d\omega =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} S_{sm}(\omega)\frac{1-\cos(k_{sm}-\omega)T}{(k_{sm}-\omega)^2}\,d\omega = \qquad (1.13)$$

$$= \frac{1}{\pi}S_{sm}(k_{sm})\int_{k_{sm}-1/T_1}^{k_{sm}+1/T_1}\frac{1-\cos(k_{sm}-\omega)T}{(k_{sm}-\omega)^2}\,d\omega + \varphi_{sm}(T) = TS_{sm}(k_{sm}) + \psi_{sm}(T)$$

Here $\varphi_{sm}(T)$ and $\psi_{sm}(T)$ are functions which remain bounded as $T \to \infty$. Analogously, for $\lambda_s - \lambda_m = ik_{sm}$ we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S'_{sm}(\omega)\left[\frac{e^{(\lambda_m-\lambda_s+i\omega)T}-1}{(\lambda_m-\lambda_s+i\omega)^2} - \frac{T}{\lambda_m-\lambda_s+i\omega}\right]d\omega = \frac{T}{2}S'_{sm}(k_{sm}) + \eta_{sm}(T) \qquad (1.14)$$

where $\eta_{sm}(T)$ is a bounded function.

We substitute (1.13) and (1.14) into (1.11) and (1.12) and we separate the terms which increase unboundedly as T grows. It is obvious that such terms may occur only in expressions corresponding to $s \leqslant 2r$. When $s > 2r$ the factor $e^{2\operatorname{Re}\lambda_s T}$ eliminates the possibility of unbounded growth since $\operatorname{Re}\lambda_s < 0$. For $s \leqslant 2r$ only the terms corresponding to $m \leqslant 2r$ may be unbounded, since the difference $\lambda_m - \lambda_s$ cannot be pure imaginary when $m > 2r$, $s \leqslant 2r$. Taking this into consideration and substituting (1.9) - (1.12) into (1.7), for $s \leqslant 2r$ ($k_{sm} = k_s - k_m$) we obtain

$$\langle x_s(T)\,\bar{x}_s(T)\rangle = \langle x_s(0)\,\bar{x}_s(0)\rangle + \mu^2 T\left\{\sum_{m=1}^{2r}[S_{sm}(k_s-k_m)\langle x_m(0)\,\bar{x}_m(0)\rangle + \right.$$

$$+\operatorname{Re} S_{sm}'(k_s-k_m)\langle x_s(0)\bar{x}_s(0)\rangle]-2\operatorname{Re} b_{ss}\langle x_s(0)\bar{x}_s(0)\rangle\}+\mu^2\theta_s(T)+\ldots \quad (1.15)$$

where $\theta_s(T)$ is a bounded function, and the terms discarded have an order of smallness higher than the second. If $T\sim\mu^{-1}$, then $\mu^2 T\sim\mu^1$ and expression (1.5) may be written to within terms of the order of μ^2 in the form

$$\mathbf{v}(T)=\mathbf{v}(0)+\mu^2 T\Gamma\mathbf{v}(0) \quad (1.16)$$

Here $\mathbf{v}(t)$ is a $2r$-dimensional vector with components $\langle|x_m(t)|^2\rangle$, Γ is a $2r\times 2r$ matrix with elements

$$\gamma_{ss}=S_{ss}(0)+\sum_{m=1}^{2r}\operatorname{Re} S_{sm}'(k_s-k_m)-2\operatorname{Re} b_{ss}, \quad \gamma_{sm}=S_{sm}(k_s-k_m) \quad (1.17)$$

We can convince ourselves that under the assumptions made the quantities $\langle x_s(T)\, x_m(T)\rangle$ for $s\neq m$, and $\langle p_{lk}(T)\,\bar{x}_s(T)\rangle$ are small quantities of the order of μ^2. If now in analogous fashion we express $\mathbf{v}(2T)$ in terms of $\mathbf{v}(T)$, then in expressions (1.10) - (1.12) these quantities yield corrections of the order of μ^3 and higher, which does not affect the final results. Thus, for integers p the quantities $\mathbf{v}(pT)$ and $\mathbf{v}[(p-1)T]$ are connected by the finite-difference relations

$$\mathbf{v}(pT)=(E+\mu^2 T\Gamma)\,\mathbf{v}[(p-1)T] \quad (1.18)$$

where E is the unit matrix.

Since the quantities $\langle|x_s(T)|^2\rangle$ are bounded for $s>2r$, a sufficient condition for the boundedness of $\langle|\mathbf{x}(T)|^2\rangle$ is the boundedness of the solutions of system (1.18) for any $\mathbf{v}(0)$. For this, as is well known, it is sufficient that the roots of the equation

$$\det\,[E+\mu^2 T\Gamma-E\eta]=0 \quad (1.19)$$

satisfy the condition $|\eta_s|<1$ $(s=1,\ldots,2r)$. Hence we can obtain an equivalent (for small μ) condition, which is that the equation

$$\det\,(\Gamma-E\rho)=0 \quad (1.20)$$

must have all its roots with negative real parts. Thus, the investigation of the stability of system (1.2) is reduced to the investigation of the roots of the characteristic equation (1.20). It is obvious that by virtue of the boundedness of matrix H^{-1} the stability conditions for systems (1.2) and (1.1) coincide.

2. Examples. 1. Let us investigate the stability of the second-order system

$$x''+2n\mu^2 x'+k^2 x+\mu\nu(t)\,x'+\mu\xi(t)\,x=0 \quad (2.1)$$

Writing it in the form of system (1.1) we obtain

$$C=\left\|\begin{matrix}0 & 1\\ k^2 & 0\end{matrix}\right\|, \qquad G(t)=\left\|\begin{matrix}0 & 0\\ -\xi(t) & -\nu(t)\end{matrix}\right\|, \qquad B=\left\|\begin{matrix}0 & 0\\ 0 & 2n\end{matrix}\right\|$$

We see that in the given case

$$H=\left\|\begin{matrix}1 & 1\\ ik & -ik\end{matrix}\right\|$$

Transforming the system to form (1.2) we have

$$A=\left\|\begin{matrix}ik & 0\\ 0 & -ik\end{matrix}\right\|, \quad P(t)=\frac{1}{2ki}\left\|\begin{matrix}-\xi(t)-ik\nu(t), & -\xi(t)+ik\nu(t)\\ \xi(t)+ik\nu(t), & \xi(t)-ik\nu(t)\end{matrix}\right\|, \quad B=\left\|\begin{matrix}n & -n\\ -n & n\end{matrix}\right\|$$

Further we find

$$S_{11}(\omega) = S_{21}(\omega) = \frac{1}{4k^2} S_\xi(\omega) + \frac{1}{4} S_\nu(\omega) + \frac{1}{2k} \operatorname{Im} S_{\xi\nu}(\omega)$$

$$S_{12}(\omega) = S_{22}(\omega) = \frac{1}{4k^2} S_\xi(\omega) + \frac{1}{4} S_\nu(\omega) - \frac{1}{2k} \operatorname{Im} S_{\xi\nu}(\omega)$$

$$\operatorname{Re} S_{11}'(\omega) = \operatorname{Re} S_{22}'(\omega) = -\frac{1}{4k^2} S_\xi(\omega) + \frac{1}{4} S_\nu(\omega)$$

$$\operatorname{Re} S_{12}'(\omega) = \frac{1}{4k^2} S_\xi(\omega) + \frac{1}{4} S_\nu(\omega) - \frac{1}{2k} \operatorname{Im} S_{\xi\nu}(\omega)$$

$$\operatorname{Re} S_{21}'(\omega) = \frac{1}{4k^2} S_\xi(\omega) + \frac{1}{4} S_\nu(\omega) + \frac{1}{2k} \operatorname{Im} S_{\xi\nu}(\omega)$$

where $S_\xi(\omega)$, $S_\nu(\omega)$, $S_{\xi\nu}(\omega)$ are the spectral densities of $\xi(t)$, $\nu(t)$ and their mutual spectral density. From formulas (1.17) we find

$$\gamma_{11} = \gamma_{22} = \frac{1}{2} S_\nu(0) + \frac{1}{4} S_\nu(2k) + \frac{1}{4k^2} S_\xi(2k) - \frac{1}{2k} \operatorname{Im} S_{\xi\nu}(2k) - 2n$$

$$\gamma_{12} = \gamma_{21} = \frac{1}{4k^2} S_\xi(2k) + \frac{1}{4} S_\nu(2k) - \frac{1}{2k} \operatorname{Im} S_{\xi\nu}(2k)$$

By setting up the usual Hurwitz conditions for Eq. (1.20), we arrive at the following stability condition:

$$n > \frac{1}{4} S_\nu(0) + \frac{1}{4} S_\nu(2k) + \frac{1}{4k^2} S_\xi(2k) - \frac{1}{k} \operatorname{Im} S_{\xi\nu}(2k) \tag{2.2}$$

For $\nu = 0$ and $S_\xi(\omega) \equiv S_0$ this condition yields $n > S_0/4k^2$, which coincides with the well-known stability condition for the case when the random parameter is a "white noise."

2. Let us consider the system of two second-order equations

$$\begin{aligned} &z_1'' + k_1^2 z_1 + \mu^2\beta_1 z_1' + \mu\xi(t)(a_{11}z_1 + a_{12}z_2) = 0 \\ &z_2'' + k_2^2 z_2 + \mu^2\beta_2 z_2' + \mu\xi(t)(a_{21}z_1 + a_{22}z_2) = 0 \end{aligned} \tag{2.3}$$

Writing system (2.3) as four first-order equations and transforming matrix C to diagonal form, we obtain

$$A = \begin{Vmatrix} ik_1 & 0 & 0 & 0 \\ 0 & -ik_1 & 0 & 0 \\ 0 & 0 & ik_2 & 0 \\ 0 & 0 & 0 & -ik_2 \end{Vmatrix}, \qquad B = \frac{1}{2} \begin{Vmatrix} \beta_1 & -\beta_1 & 0 & 0 \\ -\beta_1 & \beta_1 & 0 & 0 \\ 0 & 0 & \beta_2 & -\beta_2 \\ 0 & 0 & -\beta_2 & \beta_2 \end{Vmatrix}$$

$$P(t) = \frac{i\xi(t)}{2k_1 \cdot k_2} \begin{Vmatrix} a_{11}k_2 & a_{11}k_2 & a_{12}k_2 & a_{12}k_2 \\ -a_{11}k_2 & -a_{11}k_2 & -a_{12}k_2 & -a_{12}k_2 \\ a_{21}k_1 & a_{21}k_1 & a_{22}k_1 & a_{22}k_1 \\ -a_{21}k_1 & -a_{21}k_1 & -a_{22}k_1 & -a_{22}k_1 \end{Vmatrix}$$

In the given case the spectral densities $S_{sm}(\omega)$ are

$$S_{12} = S_{22} = -S_{12} = -S_{21} = \frac{1}{4} \frac{a_{11}^2}{k_2^2} S_\xi$$

$$S_{13} = S_{31} = S_{24} = S_{42} = -S_{13} = -S_{31} = -S_{24} = -S_{42} = \frac{1}{4} \frac{a_{12}a_{21}}{k_1 k_2} S_\xi$$

$$S_{33} = S_{44} = -S_{34} = -S_{43} = \frac{1}{4} \frac{a_{22}^2}{k_1^2} S_\xi$$

$$S'_{sm}(\omega) = -S_{sm}(\omega)$$

The elements of the matrix Γ defined by formulas (1.17) take the form:

$$\gamma_{11} = \gamma_{22} = \frac{a_{11}^2}{4k_1^2} S_\xi(2k_1) + \frac{a_{12}a_{21}}{4k_1k_2}(S_\xi^+ - S_\xi^-) - \beta_1$$

$$\gamma_{33} = \gamma_{44} = \frac{a_{22}^2}{4k_2^2} S_\xi(2k_2) + \frac{a_{12}a_{21}}{4k_1k_2}(S_\xi^+ - S_\xi^-) - \beta_2$$

$$\gamma_{12} = \gamma_{21} = -\frac{a_{11}^2}{4k_1^2} S_\xi(2k_1), \qquad \gamma_{34} = \gamma_{43} = -\frac{1}{4}\frac{a_{22}^2}{k_2^2} S_\xi(2k_2)$$

$$\gamma_{13} = \gamma_{31} = \gamma_{24} = \gamma_{42} = \frac{1}{4}\frac{a_{12}a_{21}}{k_1k_2} S_\xi^-$$

$$\gamma_{14} = \gamma_{41} = \gamma_{23} = \gamma_{32} = \frac{1}{4}\frac{a_{11}a_{21}}{k_1k_2} S_\xi^+$$

$$S_\xi^+ = S_\xi(k_1 + k_2), \qquad S_\xi^- = S_\xi(k_1 - k_2) = S_\xi(k_2 - k_1)$$

After manipulations matrix Γ is reduced to the form

$$\Gamma = \begin{Vmatrix} \Gamma_1 & \Gamma_2 \\ 0 & \Gamma_3 \end{Vmatrix}$$

where Γ_1, Γ_2, Γ_3 are second-order square matrices. Here Eq. (1.20) splits up into the two equations

$$\det(\Gamma_1 - E\rho) = 0, \qquad \det(\Gamma_3 - E\rho) = 0$$

The elements of matrices Γ_1 and Γ_3 are, respectively,

$$\gamma_{11}^{(1)} = \frac{a_{12}a_{21}}{4k_1k_2}(S_\xi^+ - S_\xi^-) - \beta_1, \qquad \gamma_{22}^{(1)} = \frac{a_{12}a_{21}}{4k_1k_2}(S_\xi^+ - S_\xi^-) - \beta_2$$

$$\gamma_{12}^{(1)} = \gamma_{21}^{(1)} = -\frac{a_{12}a_{21}}{4k_1k_2}(S_\xi^+ - S_\xi^-)$$

$$\gamma_{11}^{(3)} = \frac{1}{2}\frac{a_{22}^2}{k_2^2} S_\xi(2k_2) + \frac{a_{12}a_{21}}{4k_1k_2}(S_\xi^+ - S_\xi^-) - \beta_2$$

$$\gamma_{22}^{(3)} = \frac{1}{2}\frac{a_{11}^2}{k_1^2} S_\xi(2k_1) + \frac{a_{12}a_{21}}{4k_1k_2}(S_\xi^+ - S_\xi^-) - \beta_1$$

$$\gamma_{12}^{(3)} = \gamma_{21}^{(3)} = \frac{a_{12}a_{21}}{4k_1k_2}(S_\xi^+ + S_\xi^-)$$

Writing down the Hurwitz conditions, we obtain the following stability conditions:

$$\beta_1 > \frac{a_{11}^2}{2k_1^2} S_\xi(2k_1) + \frac{a_{12}a_{21}}{4k_1k_2}(S_\xi^+ - S_\xi^-) \tag{2.4}$$

$$\beta_2 > \frac{a_{22}^2}{2k_2^2} S_\xi(2k_2) + \frac{a_{12}a_{21}}{4k_1k_2}(S_\xi^+ - S_\xi^-)$$

$$\left[\beta_1 - \frac{a_{11}^2}{2k_1^2} S_\xi(2k_1) - \frac{a_{12}a_{21}}{4k_1k_2}(S_\xi^+ - S_\xi^-)\right] \times$$

$$\times \left[\beta_2 - \frac{a_{22}^2}{2k_2^2} S_\xi(2k_2) - \frac{a_{12}a_{21}}{4k_1 \cdot k_2}(S_\xi^+ - S_\xi^-)\right] > \frac{a_{12}^2 a_{21}^2}{16k_1^2k_2^2}(S_\xi^+ - S_\xi^-)^2$$

3. Let us consider an automatic control system containing in the loop an element with a random gain $k[1 + \mu\eta(t)]$, where η is a centered stationary random process. Having expanded the transfer function $W(p)$ into partial functions, we can write the equations of motion of the system in the form

$$z = W(p)\,u = \sum_{s=1} \frac{B_s}{p - \lambda_s}\,u, \qquad u = \mu\eta(t)\,z \tag{2.5}$$

Let us assume that among the roots of the characteristic equation there is one pair of complex roots with a small negative real part

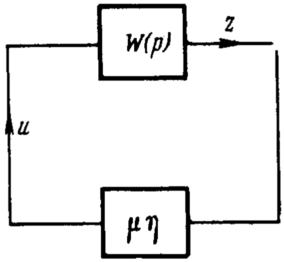

Fig. 1.

$$\lambda_{12} = -\mu^2 n \pm ik \tag{2.6}$$

The remaining roots satisfy the conditions $\operatorname{Re}\lambda_s < 0$ ($s = 3, \ldots, n$) and, moreover, $|\operatorname{Re}\lambda_s| \gg \mu^2 n$. Obviously, here $B_1 = \overline{B_2}$. Introducing the new variables

$$x_s = \frac{B_s}{p-\lambda_s} u = \frac{B_s}{p-\lambda_s}\mu\eta(t)\, z$$

we reduce Eq. (2.5) to the form

$$x_s = \lambda_s x_s + B_s \mu\eta\ (t) \sum_{s=1}^{n} x_s$$

or, with due regard to (2.6),

$$x_1^{\cdot} = -\mu^2 n x_1 + ikx_1 + \mu B_1 \eta(t) \sum_{s=1}^{n} x_s$$

$$x_2 = -\mu^2 n x_2 - ikx_2 + \mu B_2 \eta(t) \sum_{s=1}^{n} x_s$$

$$x_r^{\cdot} = \lambda_r x_r + \mu B_2 \eta(t) \sum_{s=1}^{n} x_s \qquad (r = 3, \ldots, n)$$

Determining the elements of matrix Γ, we find

$$\gamma_{11} = \gamma_{22} = 2a^2 S_\eta(0) + (a^2 + b^2) S_\eta(2k) - 2n$$

$$\gamma_{12} = \gamma_{21} = (a^2 + b^2) S_\eta(2k) \qquad a = \operatorname{Re} B_1, \qquad b = \operatorname{Im} B_1$$

Hence we obtain the stability condition

$$n > a^2 S_\eta(0) + (a^2 + b^2) S_\eta(2k) \tag{2.7}$$

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